

NEW DERIVED FROM ANOSOV DIFFEOMORPHISMS WITH PATHOLOGICAL CENTER FOLIATION.

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ABSTRACT. In this paper we focused our study on Derived From Anosov diffeomorphisms (DA diffeomorphisms) of the torus \mathbb{T}^3 , it is, an absolute partially hyperbolic diffeomorphism on \mathbb{T}^3 homotopic to an Anosov linear automorphism of the \mathbb{T}^3 . We can prove that if $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ is a volume preserving DA diffeomorphism homotopic to linear Anosov A , such that the center Lyapunov exponent satisfies $\lambda_f^c(x) > \lambda_A^c > 0$, with x belongs to a positive volume set, then the center foliation of f is non absolutely continuous. We construct a new open class U of non Anosov and volume preserving DA diffeomorphisms, satisfying the property $\lambda_f^c(x) > \lambda_A^c > 0$ for m -almost everywhere $x \in \mathbb{T}^3$. Particularly for every $f \in U$, the center foliation of f is non absolutely continuous.

1. INTRODUCTION AND STATEMENTS OF THE RESULTS

Let M be a C^∞ riemannian closed (compact, connected and boundaryless) manifold. A C^1 -diffeomorphism $f : M \rightarrow M$ is called a partially hyperbolic diffeomorphism if the tangent bundle TM admits a Df invariant tangent decomposition $TM = E^s \oplus E^c \oplus E^u$ such that all unitary vectors $v^\sigma \in E_x^\sigma, \sigma \in \{s, c, u\}$ for every $x \in M$ satisfy:

$$\|D_x f v^s\| < \|D_x f v^c\| < \|D_x f v^u\|,$$

moreover

$$\|D_x f v^s\| < 1 \text{ and } \|D_x f v^u\| > 1$$

We say that a partially hyperbolic diffeomorphism f is an absolute partially hyperbolic diffeomorphism if

$$\|D_x f v^s\| < \|D_y f v^c\| < \|D_z f v^u\|$$

for every $x, y, z \in M$ and v^s, v^c, v^u are unitary vectors in E_x^s, E_y^c, E_z^u respectively.

From now, in this paper, when we require partial hyperbolicity, we mean absolute partially hyperbolicity and all diffeomorphisms considered are at least C^1 . We go to denote by $PH_m(\mathbb{T}^3)$ the set of all partially hyperbolic diffeomorphisms which preserve the volume form m .

Date: May 12, 2015.

In partially hyperbolic context it is well known that the sub-bundles E^s, E^u , respectively the stable and unstable sub-bundles are uniquely integrable to invariant foliations $\mathcal{F}^s, \mathcal{F}^u$ respectively (see [10]). The sub-bundle E^c is not necessarily uniquely integrable to a invariant foliation \mathcal{F}^c , in fact, in [8] the authors provide examples of (non absolute) partially hyperbolic diffeomorphisms, such that E^c is not uniquely integrable.

Definition 1.1. *A partially hyperbolic diffeomorphism $f : M \rightarrow M$ is called dynamically coherent if $E^{cs} := E^c \oplus E^s$ and $E^{cu} := E^c \oplus E^u$ are uniquely integrable to invariant foliations \mathcal{F}^{cs} and \mathcal{F}^{cu} , respectively the center stable and center unstable foliations. Particularly E^c is uniquely integrable to the center foliation \mathcal{F}^c , which is obtained by the intersection $\mathcal{F}^{cs} \cap \mathcal{F}^{cu}$.*

When $M = \mathbb{T}^3$, Brin-Burago- Ivanov, in [3], shown that:

Theorem 1.2. [3] *All partially hyperbolic diffeomorphisms $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ are dynamically coherent.*

Every diffeomorphism of the torus \mathbb{T}^n induces an automorphism of the fundamental group and there exists a unique linear diffeomorphism f_* which induces the same automorphism on $\pi_1(\mathbb{T}^n)$. The diffeomorphism f_* is called linearization of f . In this paper we study relations between the center Lyapunov exponent of f and the center Lyapunov exponents of f_* under absolute continuity of the center foliation of f . The relations founded will allow construct a new open class of diffeomorphisms in $PH_m(\mathbb{T}^3)$ which the center foliation is pathological, i.e, non absolutely continuous.

Definition 1.3. *Let $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a partially hyperbolic diffeomorphism, f is called a Derived from Anosov (DA) diffeomorphism if its linearization $f_* : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ is a linear Anosov automorphism.*

By [5], given $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a partially hyperbolic diffeomorphism, then f_* is also partially hyperbolic, moreover there is a homeomorphism $h : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ that carries center leaves of f_* to corresponding center leaves of f , it is,

$$h(\mathcal{F}_{f_*}^c(x)) = \mathcal{F}_f^c(h(x)),$$

where $\mathcal{F}_g^c(y)$ is the center leaf of g through y .

Particularly the center foliation a DA diffeomorphism of the \mathbb{T}^3 is non compact.

1.1. Lyapunov Exponents. Lyapunov exponents are important constants and measure the asymptotic behavior of dynamics in tangent space level. Let $f : M \rightarrow M$ be a measure preserving diffeomorphism. Then by the Oseledec theorem, for almost every $x \in M$ and any $v \in T_x M$ the following limit exists:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|Df^n(x) \cdot v\|$$

and it is equal to one of the Lyapunov exponents of f . For a volume preserving partially hyperbolic of \mathbb{T}^3 , which is the main object of the study in this paper, we get a full Lebesgue measure subset \mathcal{R} such that for each $x \in \mathcal{R}$:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|Df^n(x) \cdot v^\sigma\| = \lambda_f^\sigma(x),$$

where $\sigma \in \{s, c, u\}$ and $v^\sigma \in E^\sigma \setminus \{0\}$.

A result of Hammerlindl-Ures states an important dichotomy between ergodicity and the center Lyapunov exponent equal to zero.

Theorem 1.4. [11] *Suppose that $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ is a conservative C^2 -DA diffeomorphism. If f is not ergodic, then the center Lyapunov exponent is zero almost everywhere.*

1.2. Absolute Continuity. It is known that the foliations \mathcal{F}^s and \mathcal{F}^u of a conservative $C^{1+\alpha}$ diffeomorphism $f : M \rightarrow M$ satisfies a property called absolute continuity. Absolute continuity is a fundamental tool in the Hopf argument in the proof of ergodicity of $C^{1+\alpha}$ conservative Anosov diffeomorphisms. Roughly speaking a foliation \mathcal{F} of M is absolutely continuous if satisfies: Given a set $Z \subset M$, such that Z intersects the leaf $\mathcal{F}(x)$ on a zero measure set of the leaf, with x along a full Lebesgue set of M , then Z is a zero measure set of M . More precisely we write:

Definition 1.5. *We say that a foliation \mathcal{F} of M is absolutely continuous if given any \mathcal{F} -foliated box B and a Lebesgue measurable set Z , such that $\text{Leb}_{\mathcal{F}(x) \cap B}(\mathcal{F}(x) \cap Z) = 0$, for m_B -almost everywhere $x \in B$, then $m_B(Z) = 0$. Here m_B denotes the Lebesgue measure on B and $\text{Leb}_{\mathcal{F}(x) \cap B}$ is the Lebesgue measure of the submanifold $\mathcal{F}(x)$ restricted to B .*

$$\text{Leb}_{\mathcal{F}(x) \cap B}((\mathcal{F}(x) \cap B) \cap Z) = 0, \quad m_B - a.e. x \in B \Rightarrow m_B(Z) = 0.$$

It means that if P is such that $m_B(P) > 0$, then there are a measurable subset $B' \subset B$, such that $m_B(B') > 0$ and $\text{Leb}_{\mathcal{F}(x) \cap B}(\mathcal{F}(x) \cap Z) > 0$ for every $x \in B'$.

The study of absolute continuity of the center foliation started with Mañé, that noted a interesting relation between absolute continuity and the center Lyapunov exponent. The Mañé's argument can be explained as the following theorem:

Theorem 1.6. *Let $f : M \rightarrow M$ be a partially hyperbolic, dynamically coherent such that $\dim(E^c) = 1$ and \mathcal{F}^c is a compact foliation. Suppose f preserves a volume form m on M , and the set of $x \in M$ such that $\lambda_f^c(x) > 0$ has positive volume. Then \mathcal{F}^c is non absolutely continuous.*

Proof. Denote by P the set of $x \in M$ such that $\lambda_f^c(x) > 0$. Consider $\Lambda_{k,l,n} = \{x \in P \mid \|Df^j(x)|E^c\| \geq e^{\frac{j}{k}}, \text{ for every } j \geq l, \text{ and } |\mathcal{F}^c(x)| < n\}$, here $|\mathcal{F}^c(x)|$ denotes the size of the center leaf $\mathcal{F}^c(x)$ through x .

We have $P = \bigcup_{k,l,n \in \mathbb{N}} \Lambda_{k,l,n}$, in particular there are k_0, l_0, n_0 such that $m(\Lambda_{k_0,l_0,n_0}) > 0$.

Supposing that \mathcal{F}^1 is an absolutely continuous foliation, there is a center leaf $\mathcal{F}^c(x)$, such that it intersects Λ_{k_0,l_0,n_0} on a positive Lebesgue measure set of the leaf. By Poincaré-recurrence Theorem, the point x can be chosen a recurrent point, particularly there is a subsequence n_k such that $f^{n_k}(x) \in \Lambda_{k_0,l_0,n_0}$, and it implies that the size $|\mathcal{F}^c(f^{n_k}(x))| < n_0$.

On the other hand, we denote

$$\alpha = \text{Leb}_{\mathcal{F}^c(x)}(\mathcal{F}^c(x) \cap \Lambda_{k_0,l_0,n_0}),$$

so, if $j \geq k_0$ we have $|\mathcal{F}(f^j(x))| \geq \alpha \cdot e^{\frac{j}{k_0}} \rightarrow +\infty$ when $j \rightarrow +\infty$, and it contradicts $|\mathcal{F}^c(f^{n_k}(x))| < n_0$ for a subsequence n_k . Consequently all one dimensional compact and absolutely continuous center foliation implies that $\lambda_f^c(x) = 0$, for m -almost everywhere $x \in M$. \square

Remark 1.7. Katok exhibits an example of a volume preserving partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ such that \mathcal{F}^c is compact, non absolutely continuous and $\lambda_f^c(x) = 0$ for m -a.e. $x \in \mathbb{T}^3$. See [6] and citations therein.

Ruelle-Wilkinson and Pesin-Hirayama generalized the Mañé argument, we can state these results in a unique theorem as following:

Theorem 1.8. [9], [15] *Consider a dynamically coherent partially hyperbolic diffeomorphism f whose center leaves are fibers of a (continuous) fiber bundle. Assume that the all center Lyapunov exponents are negative (or positive) then the conditional measures of μ on the leaves of the center foliation are atomic with $p, p \geq 1$, atoms of equal weight on each leaf.*

In the non compact case Saghin-Xia in [16] shown that:

Theorem 1.9. *Let $g \in \text{Diff}_m(\mathbb{T}^d)$ close to a linear Anosov automorphism $L : \mathbb{T}^d \rightarrow \mathbb{T}^d$ with $\lambda_L^c > 0$. If*

$$\int_{\mathbb{T}^d} \log(\|Dg|E_g^c\|) dm > \lambda_L^c,$$

then the foliation \mathcal{F}_g^c is non absolutely continuous.

In the sense of theorem 1.9 we are able to prove its generalization for DA diffeomorphisms.

Theorem 1.10 (Theorem A). *Let $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ a m -preserving DA diffeomorphism with linearization $A : \mathbb{T}^3 \rightarrow \mathbb{T}^3$. Suppose that there is a measurable set P , with $m(P) > 0$, such that $\lambda_f^c(x) > 0$ for every $x \in P$. If \mathcal{F}_f^c is absolutely continuous, then $0 \leq \lambda_f^c(x) \leq \lambda_A^c$, for m -a.e. point $x \in \mathbb{T}^3$.*

In the Anosov case, Gogolev in [4] describes completely the question of the absolute continuity of the center foliation of $C^{1+\alpha}$ conservative Anosov diffeomorphisms of \mathbb{T}^3 .

Theorem 1.11. [4] *Let L be an automorphism of \mathbb{T}^3 with three distinct Lyapunov exponents $\lambda_L^s < 0 < \lambda_L^c < \lambda_L^u$. Let U_L the open set (in the volume preserving setting) of all $C^{1+\alpha}$ volume preserving Anosov diffeomorphism homotopic to L . Let $f \in U_L$, then \mathcal{F}_f^c is absolutely continuous if, and only if, $\lambda^u(p) = \lambda^u(q)$ for every periodic points p and q .*

In [14] the authors proved study the desintegration of the volume along to \mathcal{F}_f^c , where f is a DA diffeomorphism of \mathbb{T}^3 homotopic to a liner hyperbolic automorphism A , when $\lambda_f^c \cdot \lambda_A^c < 0$.

Theorem 1.12. *Consider A a linear Anosov automorphism of \mathbb{T}^3 with three distinct Lyapunov exponents $\lambda_A^s < \lambda_A^c < \lambda_A^u$. Let $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be volume preserving DA diffeomorphism (homotopic to A). Assume that f is partially hyperbolic, volume preserving and ergodic. Also assume that $\lambda_f^c \cdot \lambda_A^c < 0$, then the disintegration of volume along center leaves of f is atomic and in fact there is just one atom per leaf.*

In this paper we treated the case $\lambda_f^c \cdot \lambda_A^c > 0$, non considered in [14]. Denote by $DA_m(\mathbb{T}^3)$ the set of all m preserving DA diffeomorphism of \mathbb{T}^3 and $\mathcal{A}(\mathbb{T}^3)$ the set of all Anosov diffeomorphism of \mathbb{T}^3 . In the case $\lambda_f^c \cdot \lambda_A^c > 0$, we can prove:

Theorem 1.13 (Theorem B). *There is an open set $U \subset DA_m(\mathbb{T}^3) \setminus \overline{\mathcal{A}(\mathbb{T}^3)}$, such that for any $f \in U$ has the same linearization A and $\lambda_f^c(x) > \lambda_A^c > 0$, for m almost everywhere $x \in \mathbb{T}^3$. Particularly \mathcal{F}_f^c is non absolutely continuous for every $f \in U$.*

The proof of Theorem B consists to combine two different types of perturbations in some steps as follows:

- (1) Firstly, using the linear maps introduced in [13], we have linear Anosov linear automorphisms, with center Lyapunov exponent arbitrarily close to zero.
- (2) After, by a small Baraviera-Bonatti perturbation (see [1]), we increase a little the center Lyapunov exponent.
- (3) Using the conservative version of Franks lemma (see [2]), we modify the stable index of a fixed point, but yet preserving in this step the increment of the center Lyapunov exponent obtained in the previous

step. The perturbation here is made carefully, such that it remains partially hyperbolic.

- (4) The neighborhood U required in the Theorem B will be an open set in $PH_m(\mathbb{T}^3)$ around the diffeomorphism obtained in the previous step, or an open ball around an stably ergodic perturbation of diffeomorphisms obtained in the previous step.

2. PROOF OF THEOREM A

Before to prove the Theorem A, let us give some ingredients necessary to the proof.

Definition 2.1. A foliation \mathcal{F} of a closed manifold M is called *quasi-isometric* if there is a constant $Q > 0$, such that in the cover level \tilde{M} we have:

$$d_{\tilde{\mathcal{W}}}(x, y) \leq Q \cdot d_{\tilde{M}}(x, y) + Q,$$

for every x, y points in the same lifted leaf $\tilde{\mathcal{W}}$, of $\tilde{\mathcal{F}}$, where $\tilde{\mathcal{F}}$ denotes the lift of \mathcal{F} on \tilde{M} . Here $d_{\tilde{\mathcal{W}}}$ denotes the riemannian metric on $\tilde{\mathcal{W}}$ and $d_{\tilde{M}}$ is a riemannian metric of \tilde{M} .

Theorem 2.2. [3], [5] Let $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a partially hyperbolic diffeomorphism, then $\mathcal{F}^s, \mathcal{F}^c$ and \mathcal{F}^u are quasi-isometric foliations.

The quasi isometry of the invariant foliations of a partially hyperbolic diffeomorphism implies some consequences of the geometry of the leaves in large scale as stated in the next lemmas.

Lemma 2.3. [5] Let $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a partially hyperbolic diffeomorphism with linearization A . For each $k \in \mathbb{Z}$ and $C > 1$ there is an $M > 0$ such that for all $x, y \in \mathbb{R}^3$,

$$\|x - y\| \geq M \Rightarrow \frac{1}{C} < \frac{\|\tilde{f}^k(x) - \tilde{f}^k(y)\|}{\|\tilde{A}^k(x) - \tilde{A}^k(y)\|} < C,$$

where $\tilde{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ denotes the lift of f to \mathbb{R}^3 .

Lemma 2.4. [12] Let $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a partially hyperbolic diffeomorphism with linearization A . For each $n \in \mathbb{Z}$ and $\varepsilon > 0$ there exists $M > 0$ such that for every x, y in the same lifted leaf of $\tilde{\mathcal{F}}^\sigma$, $\sigma \in \{s, c, u\}$ we have

$$\|x - y\| \geq M \Rightarrow (1 + \varepsilon)^{-1} e^{n\lambda_A^\sigma} \|x - y\| \leq \|\tilde{A}^n(x) - \tilde{A}^n(y)\| \leq (1 + \varepsilon) e^{n\lambda_A^\sigma} \|x - y\|,$$

where λ_A^σ is the Lyapunov exponent of corresponding to E_A^σ and $\sigma \in \{s, c, u\}$.

Combining the previous lemmas we can state.

Lemma 2.5. *Let $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a DA diffeomorphism, such that $f_* = A$ and $\lambda_A^c > 0$. Given $k = 1$, and $C = (1 + \varepsilon)$, for a small $\varepsilon > 0$, consider $M > 0$ satisfying the lemmas 2.3 and 2.4. If $x, y \in \mathbb{R}^3$ on the same center leaf of \tilde{f} , such that $\|x - y\| > M$, then*

$$\|\tilde{f}^n(x) - \tilde{f}^n(y)\| \leq (1 + \varepsilon)^{2n} e^{n\lambda_A^c} \|x - y\|, \text{ for all } n \geq 1.$$

Proof. The proof is by induction on n . If $n = 1$, by the lemma 2.3 we have

$$\|\tilde{f}(x) - \tilde{f}(y)\| \leq (1 + \varepsilon) \cdot \|\tilde{A}(x) - \tilde{A}(y)\|,$$

and by lemma 2.4

$$\|\tilde{A}(x) - \tilde{A}(y)\| \leq (1 + \varepsilon) e^{\lambda_A^c} \|x - y\|$$

combining the two last expressions we have

$$\|\tilde{f}(x) - \tilde{f}(y)\| \leq (1 + \varepsilon)^2 e^{\lambda_A^c} \|x - y\|.$$

It is important to note that $\|\tilde{f}(x) - \tilde{f}(y)\| \geq \|x - y\| \geq M$. In fact

$$\|\tilde{f}(x) - \tilde{f}(y)\| \geq (1 + \varepsilon)^{-1} \cdot \|\tilde{A}x - \tilde{A}y\|, \text{ by lemma 2.3 and}$$

$$\|\tilde{A}x - \tilde{A}y\| \geq (1 + \varepsilon)^{-1} e^{\lambda_A^c} \cdot \|x - y\|, \text{ by lemma 2.4 ,}$$

thus

$$\|\tilde{f}(x) - \tilde{f}(y)\| \geq (1 + \varepsilon)^2 e^{\lambda_A^c} \|x - y\|,$$

if $\varepsilon > 0$ is enough small, we have $\|\tilde{f}(x) - \tilde{f}(y)\| \geq \|x - y\|$. It allows us apply the argument (for $k = 1$) above to $\tilde{f}(x), \tilde{f}(y)$ in the same center leaf in \mathbb{R}^3 .

By induction suppose that $\|\tilde{f}^k(x) - \tilde{f}^k(y)\| \leq (1 + \varepsilon)^{2k} e^{k\lambda_A^c} \|x - y\|$, for some $k \geq 1$. Like described above $\|\tilde{f}^k(x) - \tilde{f}^k(y)\| \geq M$, then we apply the argument presented in the case $k = 1$ to the points $\tilde{f}^k(x), \tilde{f}^k(y)$ in the same center leaf in \mathbb{R}^3 .

Thus,

$$\|\tilde{f}^{k+1}(x) - \tilde{f}^{k+1}(y)\| = \|\tilde{f}(\tilde{f}^k(x)) - \tilde{f}(\tilde{f}^k(y))\| \leq (1 + \varepsilon)^2 e^{\lambda_A^c} \|\tilde{f}^k(x) - \tilde{f}^k(y)\|$$

since

$$\|\tilde{f}^k(x) - \tilde{f}^k(y)\| \leq (1 + \varepsilon)^{2k} e^{k\lambda_A^c} \|x - y\|,$$

we have

$$\|\tilde{f}^{k+1}(x) - \tilde{f}^{k+1}(y)\| \leq (1 + \varepsilon)^{2(k+1)} e^{(k+1)\lambda_A^c} \|x - y\|,$$

the induction is completed. □

2.1. Proof of the Theorem A.

Proof. Under the assumptions of the Theorem A, we must to prove two facts:

- (1) $\lambda_f^c(x) > 0, x \in P$ with $m(P) > 0 \Rightarrow \lambda_A^c > 0$,
- (2) $0 \leq \lambda_f^c(x) \leq \lambda_A^c$, for m a.e. $x \in \mathbb{T}^3$.

For each $\frac{1}{d} > 0$, with $d \in \mathbb{N}$ consider the set

$$P_d := \left\{ x \in P \mid \lambda_f^c(x) \geq \frac{1}{d} \right\}.$$

Since $P = \bigcup_{d=1}^{+\infty} P_d$, and $m(P) > 0$, then there exists $d > 0$ such that $m(P_d) > 0$. Now, for each $n \in \mathbb{N}$ consider

$$P_{d,n} = \{x \in P_d \mid \|Df^k(x)\| E_f^c(x) \geq e^{\frac{k}{2d}}, \forall k \geq n\},$$

then, there exists $N > 1$ such that $m(P_{d,N}) > 0$.

For each $x \in \mathbb{T}^3$, choose B_x be an \mathcal{F}_f^c -foliated box, such that x lies in the interior of B_x . Since \mathbb{T}^3 is compact, there are x_1, \dots, x_j such that $\{B_{x_i}\}_{i=1}^j$ covers \mathbb{T}^3 . Thus $m(P_{d,N} \cap B_{x_i}) > 0$ for some $1 \leq i \leq j$.

Since \mathcal{F}_f^c is an absolutely continuous foliation, there exists $p \in B_{x_i}$ such that the component of $\mathcal{F}_f^c(p)$ in B_{x_i} intersects $P_{d,N}$ in a positive riemannian measure set of the leaf. Denote this component by the center segment $[a, b]^c$.

Suppose that $\alpha \cdot |[a, b]^c| = \text{Leb}_{\mathcal{F}_f^c(p)}([a, b]^c \cap P_{d,N}), \alpha > 0$, then in the lifting $[\tilde{a}, \tilde{b}]^c$ of $[a, b]^c$ we have

$$|\tilde{f}^n([\tilde{a}, \tilde{b}]^c)| \geq \alpha \cdot e^{\frac{n}{2d}} |[\tilde{a}, \tilde{b}]^c|, n \geq N,$$

where $|[\tilde{a}, \tilde{b}]^c|$ denotes the length of the segment $[\tilde{a}, \tilde{b}]^c$. In particular $|\tilde{f}^n([\tilde{a}, \tilde{b}]^c)| \rightarrow +\infty$,

and since \mathcal{F}_f^c is quasi-isometric, we have

$$\|\tilde{f}^n(\tilde{a}) - \tilde{f}^n(\tilde{b})\| \geq \frac{1}{Q} (|\tilde{f}^n([\tilde{a}, \tilde{b}]^c)| - Q) \rightarrow +\infty. \quad (2.1)$$

Let $k, \varepsilon > 0, C > 0$ and $M > 0$ as in the lemmas 2.3 and 2.4 and consider x_n, y_n the extremes of $\tilde{f}^n([a, b]^c)$, by quasi isometry of \mathcal{F}_f^c there exists n_0 such that if $n \geq n_0$ then $\|x_n - y_n\| > M$. Then combining the lemmas 2.3 and 2.4, we have

$$(C(1 + \varepsilon))^{-1} \leq \frac{\|\tilde{f}^k(x_n) - \tilde{f}^k(y_n)\|}{e^{k\lambda_A^c} \|x_n - y_n\|} \leq C(1 + \varepsilon),$$

by the equation (2.1) we have $\|\tilde{f}^k(x_n) - \tilde{f}^k(y_n)\| \rightarrow +\infty$, follows that $e^{\lambda_A^c} > 1$, Thus $\lambda_A^c > 0$.

It remains to prove that $0 \leq \lambda_f^c \leq \lambda_A^c$.

Suppose by contradiction that $\lambda_f^c(x) > \lambda_A^c$ on a Lebesgue positive set Λ and \mathcal{F}_f^c is absolutely continuous. Choose a small $\delta > 0$ and $\varepsilon > 0$ such that

$$m(\{x \in \Lambda \mid e^{\lambda_f^c(x)} \geq (1 + 10\varepsilon)^2 e^{\lambda_A^c + \delta}\}) > 0.$$

Define

$$\Lambda_{\delta,d} = \{x \in \Lambda \mid \|Df^n(x)|E_f^c(x)\| \geq (1 + 10\varepsilon)^2 e^{\lambda_A^c + \delta}, \forall n \geq d\}.$$

Since $m(\{x \in \Lambda \mid e^{\lambda_f^c(x)} \geq (1 + 10\varepsilon)^2 e^{\lambda_A^c + \delta}\}) > 0$, then there is some d such that $m(\Lambda_{\delta,d}) > 0$.

Now for each $x \in \mathbb{T}^3$ consider B_x an open \mathcal{F}_f^c -foliated box, such that $x \in B_x$ and for each $y \in B_x$, if $[a, b]^c$ is the center segment in B_x containing y , then its lifting denoted by $[\tilde{a}, \tilde{b}]^c$ is such that $\|\tilde{a} - \tilde{b}\| > M$. Where M satisfies the lemma 2.5 with $C = (1 + \varepsilon)$ and $k = 1$. By compactness of \mathbb{T}^3 , there are B_{x_1}, \dots, B_{x_j} a finite subcover of \mathbb{T}^3 . Since \mathcal{F}_f^c is absolutely continuous, then there is $1 \leq i \leq j$ such that

$$\text{Leb}_{[a,b]^c}([a, b]^c \cap \Lambda_{\delta,d}) > 0,$$

where $[a, b]^c$ is a center connected component in B_{x_i} .

Let $\alpha > 0$ be such that $\text{Leb}_{[a,b]^c}([a, b]^c \cap \Lambda_{\delta,d}) = \alpha \cdot |[a, b]^c|$.

Thus, the length

$$|\tilde{f}^n([\tilde{a}, \tilde{b}]^c)| \geq \alpha \cdot (1 + 10\varepsilon)^{2n} e^{n(\lambda_A^c + \delta)} |[\tilde{a}, \tilde{b}]^c|, \forall n \geq d.$$

particularly using quasi isometry of the foliation \mathcal{F}_f^c we have

$$\|\tilde{f}^n(\tilde{a}) - \tilde{f}^n(\tilde{b})\| \geq \frac{\alpha}{Q} ((1 + 10\varepsilon)^{2n} e^{n(\lambda_A^c + \delta)} |[\tilde{a}, \tilde{b}]^c| - Q). \quad (2.2)$$

Applying the lemma 2.5 to $[\tilde{a}, \tilde{b}]^c$, with $C = (1 + \varepsilon)$, we obtain

$$\|\tilde{f}^n(\tilde{a}) - \tilde{f}^n(\tilde{b})\| \leq (1 + \varepsilon)^{2n} e^{n\lambda_A^c} \|\tilde{a} - \tilde{b}\|, \text{ for all } n \geq 1. \quad (2.3)$$

The inequalities (2.2) and (2.3) contradict one each other, then \mathcal{F}_f^c can not be absolutely continuous under the assumptions.

□

Corollary 2.6. *Let $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ a C^1 -volume preserving DA diffeomorphism. Suppose that there are Lebesgue measurable sets P, N , with $m(P) \cdot m(N) > 0$, such that $\lambda_f^c(x) > 0$ for every $x \in P$ and $\lambda_f^c(x) < 0$ for every $x \in N$, then \mathcal{F}_f^c is not absolutely continuous.*

Proof. Suppose that \mathcal{F}_f^c is absolutely continuous and there are Lebesgue measurable sets P, N , with $m(P) \cdot m(N) > 0$, such that $\lambda_f^c(x) > 0$ for every $x \in P$ and $\lambda_f^c(x) < 0$ for every $x \in N$.

$$\lambda_f^c(x) > 0 \text{ for every } x \in P \Rightarrow \lambda_A^c > 0,$$

$$\lambda_f^c(x) < 0 \text{ for every } x \in N \Rightarrow \lambda_A^c < 0.$$

The last implications above are contradictories, it concludes the proof. \square

Remark 2.7. The statement of the corollary above makes sense only in the C^1 setting, if f is C^r , $r \geq 2$ the ergodicity in the theorem 1.4 obstructs the existence of Lebesgue measurable sets P, N , with $m(P) \cdot m(N) > 0$, such that $\lambda_f^c(x) > 0$ for every $x \in P$ and $\lambda_f^c(x) < 0$ for every $x \in N$.

In [17] the author shown:

Theorem 2.8. [17] *Let $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ a $C^{1+\alpha}$ -DA diffeomorphism, homotopic to A , with $\lambda_A^c > 0$. Then there is a unique maximizing entropy measure μ for f , moreover (f, μ) and (A, m) are isomorphic and $\lambda_f^c(x) \geq \lambda_A^c$ for μ almost everywhere $x \in \mathbb{T}^3$.*

Relying in the previous theorem we can prove the following corollary.

Corollary 2.9. *Let $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ a $C^{1+\alpha}$ -volume preserving DA diffeomorphism, homotopic to A , with $\lambda_A^c > 0$. Suppose that m is the maximal entropy measure and \mathcal{F}_f^c is absolutely continuous, then $\lambda_f^\sigma(x) = \lambda_A^\sigma$, $\sigma \in \{s, c, u\}$ for m almost everywhere $x \in \mathbb{T}^3$.*

Proof. By the theorem 2.8 we have $\lambda_f^c(x) \geq \lambda_A^c > 0$ for m -almost everywhere $x \in \mathbb{T}^3$. On the other hand, since $\lambda_f^c(x) > 0$ for m almost everywhere $x \in \mathbb{T}^3$ and \mathcal{F}_f^c is absolutely continuous, then by Theorem A we have $\lambda_f^c \leq \lambda_A^c$ for almost everywhere $x \in \mathbb{T}^3$. So $\lambda_f^c(x) = \lambda_A^c$ for almost everywhere $x \in \mathbb{T}^3$.

By the theorem 2.8 the systems (f, m) and (A, m) are isomorphic, then

$$h_m(f) = h_m(A) = \lambda_A^c + \lambda_A^c,$$

by Pesin's formula

$$h_m(f) = \lambda_A^c + \int_{\mathbb{T}^3} \lambda_f^u dm = \lambda_A^c + \lambda_A^c = h_m(A),$$

since, by [12] we have $\lambda_f^u(x) \leq \lambda_A^u$ for m almost everywhere $x \in \mathbb{T}^3$, it jointly with the expression above imply that $m(\{x \in \mathbb{T}^3 \mid \lambda_f^u(x) < \lambda_A^u\}) = 0$. Then

$\lambda_f^u(x) = \lambda_A^u$, for m almost everywhere $x \in \mathbb{T}^3$. Consequently $\lambda_f^s(x) = \lambda_A^s$ for m almost everywhere $x \in \mathbb{T}^3$.

□

3. PROOF OF THEOREM B

For to give the proof of Theorem B, firstly we construct examples of $f \in DA_m(\mathbb{T}^3) \setminus \overline{\mathcal{A}}(\mathbb{T}^3)$, such that $\lambda_f^c > \lambda_A^c > 0$ for m -a.e., where A is the linearization of f . The open sets required will be neighborhoods of these examples or neighborhoods of perturbations of the constructed examples. For the construction we need recall some results.

Proposition 3.1. *[Conservative Franks Lemma, proposition 7.4 of [2]] Consider a conservative diffeomorphism f and a finite f -invariant set E . Assume that B is a conservative ε -perturbation of Df along E . Then for every neighbourhood V of E there is a conservative diffeomorphism g arbitrarily C^1 -close to f coinciding with f on E and out of V , and such that Dg is equal to B on E .*

Proposition 3.2. [1] *Let (M, m) be a compact manifold endowed with a C^r volume form, $r \geq 2$. Let f be a C^1 and m -preserving diffeomorphisms of M , admitting a dominated partially hyperbolic splitting $TM = E^s \oplus E^c \oplus E^u$. Then there are arbitrarily C^1 -close and m -preserving perturbation g of f , such that*

$$\int_M \log(\|Dg|E_g^c\|) dm > \int_M \log(\|Df|E_f^c\|) dm.$$

For to begin the construction, for each $n \geq 1$, as in [13], we consider $L_n : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ the Anosov automorphism the \mathbb{T}^3 induced by the matrices

$$L_n = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & -1 & n \end{bmatrix}.$$

We go to considerate the linear automorphism induced by $B_n = L_n^{-1}$. By [13] B_n have three distinct eigenvalues $\beta^u(n), \beta^c(n), \beta^s(n)$, satisfying

$$\frac{\beta^u(n)}{n} \rightarrow 1, \tag{3.1}$$

$$\beta^c(n) \rightarrow 1^+, \tag{3.2}$$

$$n \cdot \beta^s(n) \rightarrow 1, \tag{3.3}$$

calling E_n^u, E_n^c, E_n^s respectively the eigenspaces corresponding to $\beta^u(n), \beta^c(n), \beta^s(n)$, by [13] we have:

$$E_n^u \rightarrow \langle 1, 0, 0 \rangle, \quad (3.4)$$

$$E_n^c \rightarrow \langle 0, 1, 0 \rangle, \quad (3.5)$$

$$E_n^s \rightarrow \langle 0, 0, 1 \rangle, \quad (3.6)$$

where $\langle v \rangle$ denotes the real subspace spanned by v . We go to denote $\beta^c(n) = 1 + \alpha_n$, such that $\alpha_n \downarrow 0$. The center exponent $\lambda_n^c := \lambda_{B_n}^c = \log(1 + \alpha_n) \downarrow 0$.

By the proposition 3.2, and the structural stability of Anosov diffeomorphisms, consider g_n a small perturbation of L_n , an Anosov diffeomorphism such that B_n and g_n are α_n close in the C^1 topology and

$$\log(1 + \alpha_n) = \lambda_n^c < \int_{\mathbb{T}^3} \log(\|Dg_n|E_{g_n}^c\|) dm < \log(1 + 2\alpha_n). \quad (3.7)$$

Let p_n be a fixed point for g_n . Consider a system $\{V_{n_j}\}_{j=0}^{+\infty}$ of small open balls centered in p_n . The neighborhoods V_{n_j} are constructed after. Using the proposition 3.1 we can obtain a C^1 -perturbation g_{n_j} of g_n satisfying:

- (1) p_n is a fixed point for g_{n_j} ,
- (2) $g_{n_j} = g_n$ out of V_{n_j} ,
- (3) $Dg_{n_j}(p_n)|E_{g_{n_j}}^u(p_n) = Dg_n(p_n)|E_{g_n}^u(p_n)$,
- (4) $Dg_{n_j}(p_n)$ and $Dg_n(p_n)$ have the same eigenspaces and the same orientation on corresponding eigenspaces,
- (5) $\|Dg_{n_j}(p_n)|E_{g_{n_j}}^c(p_n)\| = 1 - \alpha_n$,
- (6) $Dg_{n_j}(p_n)|E_{g_{n_j}}^s(p_n)$ is taken coherently with g_{n_j} being m -preserving.

3.1. The chosen of the open system $\{V_{n_j}\}_{j=1}^{+\infty}$. Fixed $\varepsilon > 0$ a small number and $V_{n_0} = B(p_n, \varepsilon)$ the open ball centered in p_n with radius ε . Since g_n is Anosov, it is possible to choose $0 < \varepsilon_1 < \varepsilon_0$, such that, if $x \in \mathbb{T}^3 \setminus V_{n_0}$ then

$$g_n^k(x) \in B(p_n, \varepsilon_1) \Rightarrow |k| > 1,$$

we take $V_{n_1} = B(p_n, \varepsilon_1)$. By proposition 3.1 we have

$$g_n^k(x) \in B(p_n, \varepsilon_1) \Rightarrow |k| > 1.$$

Suppose that we have defined $V_{n_0}, V_{n_1}, \dots, V_{n_j}$, we define recursively $V_{n_{j+1}} = B(p_n, \varepsilon_{j+1})$, with $0 < \varepsilon_{j+1} < \varepsilon_j$, such that, if $x \in \mathbb{T}^3 \setminus V_{n_j}$ then

$$g_n^k(x) \in B(p_n, \varepsilon_{j+1}) \Rightarrow |k| > j + 1,$$

and consequently, by proposition 3.1

$$g_{n_{j+1}}^k(x) \in B(p_n, \varepsilon_{j+1}) \Rightarrow |k| > j + 1.$$

In the construction we require that $0 < \text{diam}(V_{n_j}) < \frac{1}{j+1}$ and clearly the diameters $\text{diam}(V_{n_j}) \rightarrow 0$, when $j \rightarrow +\infty$.

3.2. The diffeomorphisms g_{n_j} remains partially hyperbolic.

Lemma 3.3. *For n large, the diffeomorphisms g_{n_j} are partially hyperbolic for any $j \geq 0$.*

Proof. Each g_{n_j} are $\varepsilon_n := 100\alpha_n$ close to B_n , in the C^1 topology. In particular the matrices $Dg_{n_j}(x)$ and B_n are ε_n close, it implies that, there is a constant $C > 0$ such that the corresponding terms of the matrices B_n and $g_{n_j}(x)$ are $C\varepsilon_n$ close, for any $x \in \mathbb{T}^3$. Without loss generality, we go to consider $C = 1$.

Since $Dg_{n_j}(x)$ are ε_n close to B_n , we have:

- (1) The restriction $B_n|E_n^u$ expands by a uniform constant bigger than $\frac{n}{2}$, then $Dg_{n_j}(x)|E_n^u$ expands by a constant bigger than $\frac{n}{2}$.
- (2) The restriction $B_n|E_n^c$ are close to the identity, then $Dg_{n_j}(x)|E_n^c$ are close to the identity.
- (3) The restriction $B_n|E_n^s$ contracts by a uniform constant less than $\frac{2}{n}$, it implies that $Dg_{n_j}(x)|E_n^s$ contracts by a uniform constant less than $\frac{2}{n} + \varepsilon_n$.

The items above implies that (from canonical Jordan form) there are subspaces $F_{n_j}^u(x)$ and $F_{n_j}^s(x)$ invariant for $Dg_{n_j}(x)$, such that

$$Dg_{n_j}(x)|F_{n_j}^u(x) \text{ is uniform expanding with constant bigger than } \frac{n}{2}$$

$$Dg_{n_j}(x)|F_{n_j}^s(x) \text{ is uniform contracting with constant less than } \frac{2}{n} + \varepsilon_n,$$

Let $w_{n_j}^u(x) \in F_{n_j}^u(x) \setminus \{0\}$ and $w_{n_j}^s(x) \in F_{n_j}^s(x) \setminus \{0\}$, be unit vectors. Also, for each n , consider $\{e_n^u, e_n^c, e_n^s\}$ unitary eigenvectors of B_n .

Since $Dg_{n_j}(x)$ contracts uniformly $w_{n_j}^s(x)$ by a constant less than $\frac{2}{n} + \varepsilon_n$, then the projection of $w_{n_j}^s(x)$ on E_n^u has size of the order of $\frac{1}{n} \cdot (\frac{2}{n} + \varepsilon_n)$. A similar argument allows to claim that the projection of $w_{n_j}^s(x)$ on E_n^c has size of the order of $(\frac{2}{n} + \varepsilon_n)$. In particular when $n \rightarrow +\infty$ we have

$$\angle(E_n^s, F_{n_j}^s(x)) \rightarrow 0. \quad (3.8)$$

Analogously, taking the inverses we have

$$\angle(E_n^u, F_{n_j}^u(x)) \rightarrow 0. \quad (3.9)$$

Since E_n^u, E_n^s are one dimensional subspaces, by the equations (3.8) and (3.9) we conclude that $\dim F_{n_j}^u(x) = \dim F_{n_j}^s(x) = 1$ for every $x \in \mathbb{T}^3$, any $j \geq 1$ when n is enough large.

Fixed a constant $0 < \theta < 1$ and denote by P^σ the projection of a vector on E_n^σ , $\sigma \in \{s, c, u\}$, we have

$$\|P^u(Dg_{n_j}(x) \cdot e_n^u)\| \geq \frac{n}{2},$$

for n large it comes from (3.9), moreover

$$\|P^s(Dg_{n_j}(x) \cdot e_n^s)\| < 1 \text{ and } \|P^c(Dg_{n_j}(x) \cdot e_n^c)\| < 2,$$

it is true by (3.8) and the fact $Dg_{n_j}(x)|E_n^c$ is close to the identity.

Then for n large the cone $C_{n_j}^u(x, \theta) = \{a \cdot e_n^u + b \cdot e_n^c + k \cdot e_n^s \in T_x \mathbb{T}^3 \mid |b| + |k| \leq \theta \cdot |a|\}$ is sent in $C_{n_j}^u(g_{n_j}(x), \theta)$, by $Dg_{n_j}(x)$, it means

$$Dg_{n_j}(x) \cdot C_{n_j}^u(x, \theta) \subset C_{n_j}^u(g_{n_j}(x), \theta).$$

Analogously the inverse $[Dg_{n_j}(x)]^{-1}$ sent $C_{n_j}^s(g_{n_j}(x), \theta) = \{a \cdot e_n^u + b \cdot e_n^c + k \cdot e_n^s \in T_x \mathbb{T}^3 \mid |b| + |a| \leq \theta \cdot |k|\}$ in $C_{n_j}^s(x, \theta)$.

Define

$$C_{n_j}^{cu}(x, \theta) = \{a \cdot e_n^u + b \cdot e_n^c + k \cdot e_n^s \in T_x \mathbb{T}^3 \mid |k| \leq \theta \cdot (|a| + |b|)\}$$

$$C_{n_j}^{cs}(x, \theta) = \{a \cdot e_n^u + b \cdot e_n^c + k \cdot e_n^s \in T_x \mathbb{T}^3 \mid |a| \leq \theta \cdot (|b| + |k|)\}.$$

A similar argument to the one above allows us conclude that

$$Dg_{n_j}(x) \cdot C_{n_j}^{cu}(x, \theta) \subset C_{n_j}^{cu}(g_{n_j}(x), \theta)$$

and

$$[Dg_{n_j}(x)]^{-1} \cdot C_{n_j}^{cs}(g_{n_j}(x), \theta) \subset C_{n_j}^{cs}(x, \theta),$$

thus using the characterizations of partial hyperbolicity by cones (see [6]), g_{n_j} is partially hyperbolic for any j when n is arbitrarily large. \square

Remark 3.4. Note that g_{n_j} is a partially hyperbolic diffeomorphisms, but it is not an Anosov diffeomorphism. In fact, the dimension of the local stable manifold of g_{n_j} on p_n is equal two, on the other hand, every periodic point of its linearization B_n has one dimensional stable manifold. If g_{n_j} was Anosov, then B_n and g_{n_j} would be conjugated one each other, and consequently the dimension of the local stable manifolds of periodic points would must to coincide.

3.3. Convergences. Fixed n large for each $j \geq 0$ and $x \in \mathbb{T}^3 \setminus V_{n_j}$ define

$$N_{j+1}(x) = \min\{|k| \mid g_{n_{j+1}}^k(x) \in V_{n_{j+1}}\},$$

like the subsection 3.1 we have $N_{j+1}(x) > j + 1$, for every $x \in \mathbb{T}^3 \setminus V_{n_j}$, by the construction of invariant directions using cones, we have:

$$E_{g_{n_j}}^\sigma(x) \rightarrow E_{g_n}^\sigma(x), \sigma \in \{s, c, u\},$$

since $g_n = g_{n_j}$ out of V_{n_j} , $\text{diam}(V_{n_j}) \rightarrow 0$ and $N_j(x) \rightarrow +\infty$.

By dominated convergence, we have

$$\int_{\mathbb{T}^3 \setminus V_{n_j}} \log(\|Dg_{n_{j+1}}|E_{g_{n_{j+1}}}^c\|) dm \rightarrow \int_{\mathbb{T}^3} \log(\|Dg_n(x)\|) dm > \lambda_n^c > 0. \quad (3.10)$$

Since n is fixed $\|Dg_{n_j}(x)|E_{g_{n_j}}^c\|$ is bounded, as the diameter $\text{diam}(V_{n_j}) \rightarrow 0$, we have:

$$\int_{V_{n_j}} \log(\|Dg_{n_{j+1}}|E_{g_{n_{j+1}}}^c\|) dm \rightarrow 0. \quad (3.11)$$

Joining the equations (3.10) and (3.11) we have

$$\int_{\mathbb{T}^3} \log(\|Dg_{n_{j+1}}|E_{g_{n_{j+1}}}^c\|) dm \rightarrow \int_{\mathbb{T}^3} \log(\|Dg_n(x)\|) dm > \lambda_n^c > 0. \quad (3.12)$$

Since g_{n_j} and B_n are $100\alpha_n$ close in the C^1 topology, we have g_{n_j} is homotopic to B_n for every j when n is large enough such that $100\alpha_n < \frac{1}{5}$ and g_{n_j} being partially hyperbolic for every j as in the lemma 3.3.

3.4. Conclusion of the proof of Theorem B. Take a volume preserving partially hyperbolic diffeomorphism g_{n_j} satisfying

$$\int_{\mathbb{T}^3} \log(\|Dg_{n_{j+1}}|E_{g_{n_{j+1}}}^c\|) dm > \lambda_n^c > 0,$$

it is possible by the expression (3.12).

If $g_{n_j} \in \partial(\overline{\mathcal{A}}(\mathbb{T}^3))$, perturb g_{n_j} to a $f_n \in DA_m(\mathbb{T}^3) \setminus \overline{\mathcal{A}}(\mathbb{T}^3)$ a stably ergodic partially diffeomorphism (it is possible by results in [7]), such that $\lambda_{f_n}^c > \lambda_n^c > 0$.

Now consider $U \subset DA_m(\mathbb{T}^3) \setminus \overline{\mathcal{A}}(\mathbb{T}^3)$ a small neighborhood of f_n , if U is take a suitable neighborhood, then $\lambda_f^c > \lambda_n^c > 0$ with f homotopic to B_n for every $f \in U$.

When $g_{n_j} \in DA_m(\mathbb{T}^3) \setminus \overline{\mathcal{A}}(\mathbb{T}^3)$, we can apply the same argument above, in the case that g_{n_j} is not stably ergodic.

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